# NEGATIVITY OF THE CHERN NUMBER OF PARAMETER IDEALS

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## 1. Introduction

Let $I$ be an ideal in a Noetherian local ring $R$ and let $M$ be a finite $R$-module of dimension $d$ so that $\lambda(M/IM) < \infty$. Here $\lambda(N)$ denotes the length of an $R$-module $N$. Let $H_I(M,n) = \lambda(M/I^nM)$ denote the Hilbert function of $I$ with respect to $M$. The Hilbert function $H_I(M,n)$ for large $n$ is given by a polynomial $P_I(M,x)$ of degree $d$. It is written in the form

$$P_I(M,x) = e_0(I,M)\binom{x+d-1}{d} - e_1(I,M)\binom{x+d-2}{d-1} + \cdots + (-1)^d e_d(I,M)$$

where $e_i(I,M)$ for $i = 0,1,\ldots,d \in \mathbb{Z}$ are called the Hilbert coefficients of $I$ with respect to $M$. If $M = R$ then we write $H_I(n) = H_I(M,n), P_I(M,x) = P_I(x)$ and $e_i(I,M) = e_i(I)$ for $i = 0,1,\ldots,d$. The leading coefficient $e_0(I,M)$ is called the **multiplicity** of $I$ with respect to $M$ and the coefficient $e_1(I,M)$ is called the **Chern number** of $I$ with respect to $M$. We say that an ideal $I$ of a local ring $R$ is a **parameter ideal** for an $R$-module $M$ of dimension $d$ if $I$ is generated by $d$ elements and $\lambda(M/IM) < \infty$. If $I$ is a parameter ideal of a local ring $R$ then $R$ is Cohen-Macaulay if and only if $e_0(I) = \lambda(R/I)$. Moreover if $R$ is Cohen-Macaulay then $e_1(I) = 0$ for every parameter ideal $I$ of $R$. Vasconcelos observed that the Chern number can also be used to characterize Cohen-Macaulay property of $R$ for large classes of local rings. In the Yokohama Conference in 2008 Vasconcelos proposed several conjectures about the Chern number of filtrations of ideals [20]. One of these was

**The Negativity Conjecture (NC):** Let $I$ be a parameter ideal of a Noetherian local ring $R$. Then $e_1(I) < 0$ if and only if $R$ is not Cohen-Macaulay.

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The objective of this survey paper is to provide a glimpse into diverse techniques used to understand the Chern number by presenting solutions of NC for various classes of local rings.
In section 1, we present a solution of NC for one-dimensional modules over local rings. While doing so, we show that the Chern number of an ideal with respect to a module is always non positive.
In section 2, using Serre's difference formula for the Hilbert function and the Hilbert polynomial, we derive a formula due to Schenzel [18] for all the Hilbert coefficients of standard parameter ideals in generalized Cohen-Macaulay local rings. The Negativity Conjecture for unmixed generalized Cohen-Macaulay local rings follows from this formula.
In section 3, we prove a theorem due to Vasconcelos which asserts that if $(R, \mathfrak{m})$ is a Noetherian local ring of dimension $d \geq 2$ and it embeds into a finite maximal Cohen-Macualay $R$-module, then $R$ is Cohen-Macaulay if and only if $e_1(I) = 0$ for any parameter ideal $I$ of $R$. It follows as a cosequence that the NC is true for Noetherian local domains which are essentially of finite type over a field.
In section 4, the Chern number of parameter ideals in certain quotients of regular local rings is calculated explicitly. As a consequence a solution of NC is given for such rings. The proof illustrates the use of Eagon-Northcott complex for calculation of the Chern number.
Recall that a Noetherian local ring $(R, \mathfrak{m})$ is called **unmixed** if for each associated prime $\mathfrak{p}$ of the $\mathfrak{m}$-adic completion $\hat{R}$, $\dim \hat{R}/\mathfrak{p} = \dim R$.
In section 5, we present an example of a local ring which is not unmixed in which the Chern number of some parameter ideal is zero. The Negativity Conjecture has recently been settled by Ghezzi, Goto, Hong, Phuong and Vasconcelos in [3] for all unmixed local rings. We present their solution in section 5.

## 2. Solution of NC for 1-dimensional modules and nonpositivity of $e_1(I, M)$

In this section we obtain a formula for the Chern number of a parameter ideal for 1-dimensional finite modules over a Noetherian local ring. This generalizes a result of Goto-Nishida [6]. We also show that that Chern number of a parameter ideal in a local ring $R$ with respect to a finite $R$-module is nonpositive.

**Theorem 1** (Goto-Nishida, [6]). *Let $(R, \mathfrak{m})$ be a Noetherian ring and $M$ be a finite $R$-module with $\dim M = 1$. If $(a)$ is a parameter ideal for $M$ then $e_1((a), M) = -\lambda(H^0_{\mathfrak{m}}(M))$.*

*Proof:* Let $N = H^0_{\mathfrak{m}}(M)$ and $\overline{M} = M/N$. Notice that $H^0_{\mathfrak{m}}(\overline{M}) = 0$ and $\dim \overline{M} = \dim M = 1$, which implies depth $\overline{M} = 1$. Thus $\overline{M}$ is Cohen-Macaulay $R$-module. Consider the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow \overline{M} \longrightarrow 0.$$

By taking tensor product with $R/(a)^n$ we get the exact sequence for all $n \geq 1$,

$$0 \longrightarrow \ker \phi \longrightarrow N/a^nN \xrightarrow{\phi} M/a^nM \longrightarrow \overline{M}/a^n\overline{M} \longrightarrow 0. \tag{1}$$

For $n >> 0$ we have $(a)^n N \subseteq \mathfrak{m}^n H^0_{\mathfrak{m}}(M) = 0$. Thus

$$\begin{aligned}
\lambda(\overline{M}/a^n\overline{M}) &= \lambda(M/a^nM + N) \\
&= \lambda(M/a^nM) - \lambda\left(\frac{a^nM+N}{a^nM}\right) \\
&= \lambda(M/a^nM) - \lambda\left(\frac{N}{a^nM \cap N}\right) \\
&= \lambda(M/a^nM) - \lambda(N).
\end{aligned} \tag{2}$$

From (1) and (3), we get that for all large $n$,

$$\lambda(\ker \phi) = \lambda(N) - \lambda(M/a^nM) + \lambda(\overline{M}/a^n\overline{M}) = 0$$

which gives $\ker \phi = 0$. Thus we get the following exact sequence for large $n$,

$$0 \longrightarrow N \longrightarrow M/a^nM \longrightarrow \overline{M}/a^n\overline{M} \longrightarrow 0.$$

Hence we have $\lambda(N)+\lambda(\overline{M}/a^n\overline{M}) = \lambda(M/a^nM)$. Since $\overline{M}$ is Cohen-Macaulay,

$$\lambda(\overline{M}/a^n\overline{M}) = e_0((a^n), \overline{M}) = e_0((a), \overline{M})n = e_0((a), M)n.$$

Also for large $n$, $\lambda(M/a^nM) = ne_0((a), M) - e_1((a), M)$. Therefore

$$e_1((a), M) = -\lambda(H^0_{\mathfrak{m}}(M)).$$

$\square$

**Proposition 2.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $M$ be a finite $R$-module with $\dim M = 1$. Let $a$ be a parameter for $M$. Then $e_1((a), M) < 0$ if and only if $M$ is not a Cohen-Macaulay module.*

*Proof:* Let $M$ be not Cohen-Macaulay. Then $H^0_{\mathfrak{m}}(M) \neq 0$. By Theorem 1, $e_1((a), M) = -\lambda(H^0_{\mathfrak{m}}(M)) < 0$. The converse is well known [1, Theorem 1.1.8]. $\square$

**Theorem 3** (Mandal-Singh-Verma, [12]). *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $M$ be a finite $R$-module with $\dim M = d$. Let $J$ be an ideal generated by a system of parameters for $M$. Then*

$$e_1(J, M) \leq 0.$$

*Proof:* Apply induction on $d$. The case $d = 1$ is already proved. Suppose $d = 2$. Let $J = (x, y)$ where $x, y$ is a superficial sequence for $J$ with respect to $M$. Consider the exact sequence

$$0 \longrightarrow M/(0 :_M x) \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0.$$

Applying $H^0_{\mathfrak{m}}(.)$ we get

$$0 \longrightarrow H^0_{\mathfrak{m}}(M/(0 :_M x)) \xrightarrow{x} H^0_{\mathfrak{m}}(M) \xrightarrow{g} H^0_{\mathfrak{m}}(M/xM) \longrightarrow C \longrightarrow 0 \tag{3}$$

where $C = \operatorname{coker} g$. Consider the exact sequence

$$0 \longrightarrow (0 :_M x) \longrightarrow M \longrightarrow M/(0 :_M x) \longrightarrow 0.$$

Applying $H^0_{\mathfrak{m}}(.)$ to the exact sequence we get

$$0 \longrightarrow H^0_{\mathfrak{m}}(0 :_M x) \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow H^0_{\mathfrak{m}}(M/(0 :_M x)) \longrightarrow 0.$$

Since $H^0_{\mathfrak{m}}(0 :_M x) = (0 :_M x)$, we have

$$\lambda(0 :_M x) = \lambda(H^0_{\mathfrak{m}}(M)) - \lambda(H^0_{\mathfrak{m}}(M/(0 :_M x))).$$

Subtracting $\lambda(H^0_{\mathfrak{m}}(M/xM))$ from both sides of the above equation we get

$$\begin{aligned} & \lambda(0 :_M x) - \lambda(H^0_{\mathfrak{m}}(M/xM)) \\ = \; & \lambda(H^0_{\mathfrak{m}}(M)) - \lambda(H^0_{\mathfrak{m}}(M/xM)) - \lambda(H^0_{\mathfrak{m}}(M/(0 :_M x))). \end{aligned}$$

From the exact sequence (3) we get

$$\lambda(H^0_{\mathfrak{m}}(M/(0 :_M x))) - \lambda(H^0_{\mathfrak{m}}(M)) + \lambda(H^0_{\mathfrak{m}}(M/xM)) = \lambda(C).$$

Therefore we have $\lambda(0 :_M x) - \lambda(H^0_{\mathfrak{m}}(M/xM)) = -\lambda(C)$. By a module theoretic version of [13, Theorem 70] we get

$$e_1(\overline{J}, \overline{M}) = e_1(J, M) - \lambda(0 :_M x).$$

By Proposition 1, $e_1(\overline{J}, \overline{M}) = -\lambda(H^0_{\mathfrak{m}}(M/xM))$. Therefore

$$e_1(J, M) = \lambda(0 :_M x) - \lambda(H^0_{\mathfrak{m}}(M/xM)) = -\lambda(C) \leq 0.$$

Let $d \geq 3$ and $a \in J$ be superficial for $J$ with respect to $M$. Since $e_1(J, M) = e_1(J/(a), M/aM)$, we are done by induction. □

## 3. Hilbert polynomial of standard system of parameters in generalized Cohen-Macaulay local rings

A local ring $(R, \mathfrak{m})$ of dimension $d$ is said to be generalized Cohen-Macaulay local ring if $\lambda(H^i_{\mathfrak{m}}(R)) < \infty$ for $i = 0, 1, \ldots d-1$. A system of parameters $a_1, \ldots, a_d$ is called a **standard system of parameters** for $R$ if for $\mathfrak{q} = (a_1, a_2, \ldots, a_d)$,

$$\lambda(R/\mathfrak{q}) - e_0(\mathfrak{q}) = \sum_{i=0}^{d-1} \binom{d-1}{i} \lambda(H^i_{\mathfrak{m}}(R)).$$

A local ring $R$ is called **Buchsbaum** if for every parameter ideal $\mathfrak{q}$ of $R$, the difference $\lambda(R/\mathfrak{q}) - e_0(\mathfrak{q})$ is independent of $\mathfrak{q}$. In a Buchsbaum local ring, every parameter ideal is standard. Moreover, a Buchsbaum ring is generalized Cohen-Macaulay.

In this section we will give a new proof of a formula of Schenzel [18] for the coefficients of the Hilbert polynomial of a parameter ideal generated by a standard system of parameters. This is achieved via an application of Serre's formula [1] for the difference of Hilbert polynomial and Hilbert function of the associated graded ring $G(I)$ in terms of its local cohomology modules.

**Example 4.** Let $k$ be a field and $S = k[|x,y|]$. Let $R = S/(x)\cap(x^3,y)$. Then $R$ is a 1-dimensional generalized Cohen-Macaulay but not Buchsbaum. We prove that $(y)$ is a standard system of parameters of $R$. By associativity formula we have

$$e_0(y,R) = \sum_{P\in \operatorname{Ass} R, \dim R = \dim R/P} e_0(y, R/P)\lambda(R_P) = e_0(y, k[|y|]) = 1.$$

Note that $\lambda(R/yR) = \lambda(k[|x,y|]/(x^3,xy,y)) = 3$. Hence $\lambda(R/yR) - e_0(y,R) = 2$. On the other hand

$$H^0_{\mathfrak{m}}(R) = \frac{[(x):\mathfrak{m}^\infty]\cap[(x^3,y):\mathfrak{m}^\infty]}{(x^3,xy)} = \frac{(x)}{(x^3,xy)}.$$

Thus $\lambda(H^0_{\mathfrak{m}}(R)) = 2$. Hence $y$ is a standard system of parameter of the generalized Cohen-Macaulay ring $R$.

**Theorem 5.** *Let $(R,\mathfrak{m})$ be a generalized Cohen-Macaulay local ring and $\mathfrak{q}$ be an ideal generated by a standard system of parameters of $R$. Let $G(\mathfrak{q}) = \bigoplus_{n\geq 0}\mathfrak{q}^n/\mathfrak{q}^{n+1}$ be the associated graded ring. Then $\lambda(H^i_M(G(\mathfrak{q}))) < \infty$ for $i = 0,\ldots,d-1$ where $M$ is the maximal homogeneous ideal of $G(\mathfrak{q})$*

**Theorem 6** (Goto, [5])**.** *Let $(R,\mathfrak{m})$ be a generalized Cohen-Macaulay local ring of dimension $d$. Let $I$ be a standard parameter ideal and $G(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ be the associated graded ring of $I$. Then*

*(i)* $[H^i_M(G(I))]_n = (0)$ $(n \neq -i)$ *and* $[H^i_M(G(I))]_{-i} = H^i_{\mathfrak{m}}(R)$ *for all* $0 \leq i < d$.

*(ii)* $[H^d_M(G(I))]_n = (0)$ *for* $n > -d$.

**Theorem 7** (Schenzel, [18])**.** *Let $(R,\mathfrak{m})$ be a generalized Cohen-Macaulay ring of dimension $d$ and $I$ be a standard parameter ideal. Then for $i = 0,1,\ldots,d-1$,*

$$e_{d-i} = (-1)^{d-i}\sum_{j=0}^{i}\binom{i-1}{j-1}\lambda(H^j_{\mathfrak{m}}(R)).$$

*Proof:* Use induction on $i$. By Serre's difference formula for the graded ring $G(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ with the maximal homogeneous ideal $M = \mathfrak{m}G + G_+$:

$$H^0_I(n) - P^0_I(n) = \sum_{i=0}^{d}(-1)^i\lambda((H^i_M G(I))_n). \tag{4}$$

where $H^0_I(n) = \lambda(I^n/I^{n+1})$ is the Hilbert function of $G(I)$ and $P^0_I(x)$ is the corresponding Hilbert polynomial written as follows

$$P^0_I(x) = e_0\binom{x+d-1}{d-1} - e_1\binom{x+d-2}{d-2} + \cdots + (-1)^{d-1}e_{d-1}.$$

Let us first prove the case $i=0$. Notice that by Theorem 6 we have $H_I^0(n) = P_I^0(n)$ for $n \geq 1$. In equation (4) substituting $n = 0, 1, \ldots, n-1$ and adding both sides we get

$$\begin{aligned} & \sum_{i=0}^{n-1} \lambda(I^i/I^{i+1}) - \left[e_0\binom{n+d-1}{d} - e_1\binom{n+d-2}{d-1} + \cdots + (-1)^{d-1}ne_{d-1}\right] \\ = \;& \lambda(H^0_{\mathfrak{m}}(R)) \end{aligned}$$

which implies $(-1)^d e_d = \lambda(H^0_{\mathfrak{m}}(R))$. Next assume that the result is true for $i-1$ and we prove it for $i$. Substituting $n=-i$ in equation (4) we get

$$\begin{aligned} & (-1)^{d-i}e_{d-i}(-1)^{i-1} + \sum_{k=1}^{i-1}(-1)^{d-i+k}e_{d-i+k}(-1)^{i-k-1}\binom{i-1}{k} \\ = \;& (-1)^{i-1}\lambda(H^i_{\mathfrak{m}}(R)). \end{aligned}$$

Substituting the expressions for $e_{d-1}, \ldots, e_{d-i+1}$ in the above equation to get

$$\begin{aligned} & (-1)^{d-i}e_{d-i}(-1)^{i-1} + \sum_{k=1}^{i-1}(-1)^{i-k-1}\binom{i-1}{k}\left[\sum_{j=0}^{i-k}\binom{i-k-1}{j-1}\lambda(H^j_{\mathfrak{m}}(R))\right] \\ = \;& (-1)^{i-1}\lambda(H^i_{\mathfrak{m}}(R)) \end{aligned}$$

From the above equation coefficient of $\lambda(H^{i-k}_{\mathfrak{m}}(R))$ for $1 \leq k \leq i$ is

$$\begin{aligned} &= (-1)^{i-1}\sum_{l=1}^{k}(-1)^l\binom{i-1}{l}\binom{i-l-1}{i-k-1} \\ &= (-1)^{i-1}\sum_{l=1}^{k}(-1)^l\binom{i-1}{i-k-1}\binom{k}{l} \\ &= (-1)^i\binom{i-1}{i-k-1} \end{aligned}$$

Thus we have

$$\begin{aligned} & (-1)^{d-i}e_{d-i}(-1)^{i-1} \\ = \;& (-1)^{i-1}\left[\lambda(H^i_{\mathfrak{m}}(R)) + \sum_{k=1}^{i}\binom{i-1}{i-k-1}\lambda(H^{i-k}_{\mathfrak{m}}(R))\right] \\ = \;& (-1)^{i-1}\left[\lambda(H^i_{\mathfrak{m}}(R)) + \sum_{j=0}^{i-1}\binom{i-1}{j-1}\lambda(H^j_{\mathfrak{m}}(R))\right] \end{aligned}$$

Hence we get

$$(-1)^{d-i}e_{d-i} = \sum_{j=0}^{i}\binom{i-1}{j-1}\lambda(H^j_{\mathfrak{m}}(R)). \;\square$$

The above formula yields a characterization of depth in terms of vanishing of Hilbert coefficients in unmixed generalized Cohen-Macaulay local rings. In particular it solves the NC for such rings.

**Corollary 8.** *Let $(R, \mathfrak{m})$ be an unmixed generalized Cohen-Macaulay local ring of positive dimension $d$ and let $I$ be a standard parameter ideal. Fix any $i = 1, 2, \ldots, d$. Then*

$$\operatorname{depth} R \geq d - i + 1 \quad \textit{if and only if} \;\; e_i(I) = 0.$$

*Proof:* If $\operatorname{depth} R \geq d - i + 1$ then $H^j_{\mathfrak{m}}(R) = 0$ for $j = 0, \ldots, d - i$. Hence by Theorem 7 we get $e_i = 0$. Conversely if $e_i = 0$ then $H^j_{\mathfrak{m}}(R) = 0$ for $j = 1, \ldots, d-i$ and since $R$ is unmixed $H^0_{\mathfrak{m}}(R) = 0$. Thus $\operatorname{depth} R \geq d - i + 1$.

□

## 4. Solution of NC for local rings of affine domains over a field.

Vasconcelos took the first step for the solution of NC [20]. He proved that if $R$ is a non Cohen-Macaulay Noetherian local domain which is essentially of finite type over a field then the Chern number of any parameter ideal in $R$ is negative. The key idea in the proof is to embed the local domain in finite maximal Cohen-Macaulay module and use the following theorem.

**Theorem 9 (Vasconcelos).** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$. Suppose there is an embedding of $R$-modules:*

$$0 \longrightarrow R \longrightarrow E \longrightarrow C \longrightarrow 0,$$

*where $E$ is a finitely generated maximal Cohen-Macaulay $R$-module and $C = E/R$. If $R$ is not Cohen-Macaulay, then $e_1(J) < 0$ for any parameter ideal $J$.*

*Proof:* We may assume that the residue field of $R$ is infinite. Notice that $\operatorname{depth} R \geq 1$. We prove the theorem by induction on $d$. Let $d = 2$, let $J$ be a parameter ideal of $R$. If $R$ is not Cohen-Macaulay, then by depth lemma we have $\operatorname{depth} C = 0$.
Let $J = (x, y)$ where we may assume that $x$ is a superficial element for $J$ and $x$ does not belong to any non-maximal associated prime of $C$. Tensoring the exact sequence above by $R/(x)$, we get the exact sequence

$$0 \longrightarrow T = \operatorname{Tor}_1^R(R/(x), C) \longrightarrow R/(x) \longrightarrow E/xE \longrightarrow C/xC \longrightarrow 0,$$

where $T$ is a nonzero module of finite length. Denote by $S$ the image of $R' = R/(x)$ in $E/xE$. Note that $S$ is a Cohen-Macaulay ring of dimension 1. By the Artin-Rees Lemma, for $n >> 0$, $T \cap (y^n)R' = 0$, and therefore from the diagram

$$\begin{tikzcd}
0 \arrow[r] & T \cap (y^n)R' \arrow[r] \arrow[d] & (y^n)R' \arrow[r] \arrow[d] & (y^n)S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T \arrow[r] & R' \arrow[r] & S \arrow[r] & 0
\end{tikzcd}$$

using Snake Lemma, for large $n$,

$$\lambda(R'/y^nR') = \lambda(T) + \lambda(S/y^nS).$$

Hence comparing the coefficients of the Hilbert polynomial from both sides we get

$$e_0n - e_1 = e_0(yS)n + \lambda(T).$$

Therefore

$$e_1(J) = -\lambda(T) < 0. \tag{5}$$

Assume now that $d \geq 3$ and let $x$ be a superficial element for $J$, and the modules $E$ and $C$. In the exact sequence

$$0 \longrightarrow T = \mathrm{Tor}_1^R(R/(x), C) \longrightarrow R' = R/(x) \longrightarrow E/xE \longrightarrow C/xC \longrightarrow 0, \tag{6}$$

$T$ is either zero, and we would go on with the induction procedure, or $T$ is a nonzero module with finite support.

If $T \neq 0$ in the exact sequence (6), we have $e_1(JR') = e_1(JS)$. By the induction argument, it suffices to prove that $S$ is not Cohen-Macaulay.
We may assume that $R$ is a complete local ring. Since $R$ is embedded in a maximal Cohen-Macaulay module, any associated prime of $R$ is an associated prime of $E$ and therefore it is equidimensional. Consider the exact sequences

$$0 \longrightarrow T \longrightarrow R' \longrightarrow S = R'/T \longrightarrow 0,$$

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R' \longrightarrow 0.$$

From the first sequence, we get the exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(T) = T \longrightarrow H^0_{\mathfrak{m}}(R') \longrightarrow H^0_{\mathfrak{m}}(S) = 0,$$

since $H^i_{\mathfrak{m}}(T) = 0$ for $i > 0$ and $S$ is Cohen-Macaulay of dimension $\geq 2$; one also has $H^1_{\mathfrak{m}}(S) = H^1_{\mathfrak{m}}(R') = 0$. From the second sequence, since the associated primes of $R$ have dimension $d$, $H^1_{\mathfrak{m}}(R)$ is a finitely generated $R$-module. Finally, by Nakayama Lemma $H^1_{\mathfrak{m}}(R) = 0$, and therefore $T = H^0_{\mathfrak{m}}(R') = 0$. Hence $R/xR \cong S$. Since $S$ is Cohen-Macaulay, $R'$ is Cohen-Macaulay. Hence $R$ is Cohen-Macaulay. This is a contradiction.

□

In order to prove NC for affine local domains, we need the concept of balanced Cohen-Macaulay module. We recall these concepts from [1]. An $R$-module $M$ over a local ring $R$ is called a **big Cohen-Macaulay** module if there is a system of parameters $x$ for $R$ which is $M$-regular. Hochster proved that if $R$ contains a field then it has a big Cohen-Macaulay module. The module $M$ is called a **balanced big Cohen-Macaulay module** if every system of parameters in $R$ is $M$-regular. Balanced big Cohen-Macaulay modules have

many properties in common with finite modules. For example, their set of associated primes is finite. Griffith [8, Theorem 3.1] and [9, Proposition 1.4] showed that if $R$ is a complete local domain then it has a countably generated balanced big Cohen-Macaulay module.
Now using the above technique the following theorem is proved by embedding $R$ into a countably generated balanced big Cohen-Macaulay module.

**Theorem 10.** *Let $(R, \mathfrak{m})$ be a Noetherian local domain essentially of finite type over a field. If $R$ is not Cohen-Macaulay, then $e_1(J) < 0$ for any parameter ideal $J$.*

*Proof:* Let $A$ be the integral closure of $R$ and $\widehat{R}$ be its completion. Tensor the embedding $R \subset A$ to obtain the embedding

$$0 \longrightarrow \widehat{R} \longrightarrow \widehat{R} \otimes_R A = \widehat{A}.$$

From the properties of pseudo-geometric local rings [14, Section 37], $\widehat{A}$ is a reduced semi-local ring with a decomposition

$$\widehat{A} = A_1 \times \cdots \times A_r,$$

where each $A_i$ is a complete local domain, of dimension $\dim R$ and finite over $\widehat{R}$. For each $A_i$ we make use of [8, Theorem 3.1] and [9, Proposition 1.4] and pick a countably generated balanced big Cohen-Macaulay $A_i$-module and therefore $\widehat{R}$-module. Collecting the $E_i$ we have an embedding

$$\widehat{R} \longrightarrow A_1 \times \cdots \times A_r \longrightarrow E = E_1 \oplus \cdots \oplus E_r.$$

As $E$ is a countably generated balanced big Cohen-Macaulay $\widehat{R}$-module, the argument above shows if $\widehat{R}$ is not Cohen-Macaulay then $e_1(J) = e_1(J\widehat{R}) < 0$.

□

Ghezzi, Hong and Vasconcelos in [4] have proved the conjecture for universally catenarian domains and domains that are homomorphic images of Cohen-Macualay rings.

**Theorem 11.** *If $(R, \mathfrak{m})$ is Noetherian domain of dimension $d \geq 2$, which is a homomorphic image of a Cohen-Macaulay Noetherian ring and if $R$ is not Cohen-Macaulay, then $e_1(J) < 0$ for any parameter ideal $J$.*

**Theorem 12.** *Let $(R, \mathfrak{m})$ be a universally catenary integral domain containing a field. If $R$ is not Cohen-Macaulay, then $e_1(J) < 0$ for any parameter ideal $J$.*

## 5. Hilbert polynomial of parameter ideals in quotients of regular local rings

In this section we show that the Chern number is negative for parameter ideals in certain unmixed quotients of regular local rings and in some cases it

is independent of the choice of the parameter ideal by explicitly finding the Hilbert polynomial of all parameter ideals.

L. Ghezzi, J. Hong and W. Vasconcelos [4] calculated the Chern number of any parameter ideal in certain quotients of regular local rings of dimension four. We recall their result first.

**Example 13.** Let $(S, \mathfrak{m})$ be a four dimensional regular local ring with $S/\mathfrak{m}$ infinite. Let $P_1, P_2, \ldots, P_r$ be a family of height two prime ideals of $S$ so that for $i \neq j, P_i + P_j$ is $\mathfrak{m}$-primary. Put $R = S/\cap_{i=1}^r P_i$. Let $J$ be an $\mathfrak{m}$-primary parameter ideal of $R$. Let $L = [\oplus_{i=1}^r S/P_i]/R$. If $J \subseteq \operatorname{ann} L$ then $e_1(J) = -\lambda(L)$ and $e_2(J) = 0$.

**Lemma 14.** *Let $(S, \mathfrak{n})$ be an $r$-dimensional regular local ring. Let $I$ be a height $h$ ideal of $S$. Suppose $a_1, \ldots, a_d \in S$ such that $(a_1 + I, \ldots, a_d + I)$ is a system of parameters in $S/I$. Then $a_1, \ldots, a_d$ is a regular sequence in $S$.*

*Proof:* Let $J = (a_1, \ldots, a_d)$. Then $\lambda(S/I \otimes_S S/J) < \infty$. Hence by Serre's theorem [15, Theorem 3, Chapter 5]

$$\dim S/I + \dim S/J \leq \dim S = r.$$

As $S$ is regular, it is catenary. Thus $d + r - \operatorname{ht} J \leq r$. Therefore $d \leq \operatorname{ht} J \leq d$. Hence $\operatorname{ht} J = d$ and consequently $a_1, \ldots, a_d$ is an $S$-regular sequence.

$\square$

**Lemma 15.** *Let $(S, \mathfrak{n})$ be an $r$-dimensional regular Noetherian ring. Let $I$ be a height $h$ Cohen-Macaulay ideal of $S$. Suppose $J = (a_1, \ldots, a_d)$ such that $(a_1 + I, \ldots, a_d + I)$ is a system of parameters in $S/I$. Then for all $j, n \geq 1$*

$$\operatorname{Tor}_j^S(S/J^n, S/I) = 0.$$

*Proof:* We will apply induction on $n$. Let $n = 1$. As $J$ is a complete intersection, the Koszul complex $K(\underline{a})$ of the sequence $\underline{a} = a_1, a_2, \ldots, a_d$

$$K(\underline{a}) : 0 \longrightarrow S \longrightarrow S^d \longrightarrow S^{\binom{d}{2}} \longrightarrow \ldots \longrightarrow S^d \longrightarrow S \longrightarrow S/J \longrightarrow 0$$

gives a free resolution of $S/J$. Tensoring the above complex with $R := S/I$ we get

$$K(\underline{a}, S/I) : 0 \longrightarrow R \longrightarrow R^d \longrightarrow R^{\binom{d}{2}} \longrightarrow \ldots \longrightarrow R^d \longrightarrow R \longrightarrow R/K \longrightarrow 0$$

which is the Koszul complex of $JR = K$. As $K$ is generated by an $R$-regular sequence, the above is a free resolution of $R/K$. Hence $\operatorname{Tor}_j^S(S/J, S/I) = 0$ for all $j \geq 1$. Since $J$ is generated by a regular sequence, $J^n/J^{n+1}$ is a free $S/J$-module. Consider the exact sequence

$$0 \longrightarrow J^n/J^{n+1} \longrightarrow S/J^{n+1} \longrightarrow S/J^n \longrightarrow 0.$$

This gives rise to the long exact sequence

$$\cdots \longrightarrow \operatorname{Tor}_j^S(J^n/J^{n+1}, S/I) \longrightarrow \operatorname{Tor}_j^S(S/J^{n+1}, S/I) \longrightarrow \operatorname{Tor}_j^S(S/J^n, S/I) \longrightarrow$$

By induction on $n$, it follows that $\operatorname{Tor}_j^S(S/J^{n+1}, S/I) = 0$ for all $j \geq 1$.

□

**Lemma 16.** *Let $J = (a_1, \ldots, a_d)$ be a complete intersection of height $d$ in a regular local ring $(R, \mathfrak{m})$. Let $L$ be an $R$-module of finite length. Then $\mathcal{R}(J) \otimes_R L$ is a finite $\mathcal{R}(J)$-module of dimension $d$ and*

$$\operatorname{Supp}(\mathcal{R}(J) \otimes_R L) = V(\mathfrak{m}\mathcal{R}(J)).$$

**Lemma 17.** *Let $S$ be a Noetherian local ring, $a_1, \ldots, a_d$ be a regular sequence and $J = (a_1, \ldots, a_d)$. Let $L$ be an $S$-module of finite length. If $J \subseteq \operatorname{ann} L$ then for all $n \geq 1$*

$$\lambda(\operatorname{Tor}_1(L, S/J^n)) = \binom{n+d-1}{d-1} \lambda(L).$$

*Proof:* By [10, Example 10] for any $n > 0$, $J^n$ is generated by the maximal minors of the $n \times (n+d-1)$ matrix $A$ where

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_d & 0 & 0 & \cdots & 0 \\ 0 & a_1 & a_2 & \cdots & a_{d-1} & a_d & 0 & \cdots & 0 \\ 0 & 0 & a_1 & \cdots & a_{d-2} & a_{d-1} & a_d & \cdots & 0 \\ 0 & 0 & 0 & \cdots & a_1 & a_2 & a_3 & \cdots & a_d \end{pmatrix}$$

By Eagon-Northcott [2, Theorem 2], the minimal free resolution of $S/J^n$ is given by

$$0 \longrightarrow S^{\beta_d} \longrightarrow S^{\beta_{d-1}} \longrightarrow \cdots \longrightarrow S^{\beta_1} \longrightarrow S \longrightarrow S/J^n \longrightarrow 0 \tag{7}$$

where the Betti numbers of $S/J^n$ are given by

$$\beta_i^S(S/J^n) = \binom{n+d-1}{d-i}\binom{n+i-2}{i-1}, \qquad 1 \leq i \leq k.$$

Taking tensor product of (7) with $L$ we get the following complex

$$0 \longrightarrow L^{\beta_d} \longrightarrow L^{\beta_{d-1}} \longrightarrow \cdots \longrightarrow L^{\beta_1} \longrightarrow L \longrightarrow L/J^nL \longrightarrow 0.$$

Since $J \subseteq \operatorname{ann} L$, the maps in the above complex are zero. Hence

$$\lambda(\operatorname{Tor}_1(L, S/J^n)) = \beta_1 \lambda(L) = \binom{n+d-1}{d-1} \lambda(L).$$

□

**Theorem 18** (Mandal-Verma, [11]). *Let $(S, \mathfrak{n})$ be a regular local ring of dimension $r$ and $I_1, \dots, I_g$ be Cohen-Macaulay ideals of height $h$ which satisfy the condition: $I_i + I_j$ is $\mathfrak{n}$-primary for $i \neq j$. Let $R = S/I_1 \cap \dots \cap I_g$ and $d = \dim R \geq 2$. Let $a_1, \dots, a_d \in S$ such that their images in $R$ form a system of parameters. Let $J = (a_1, \dots, a_d)$, $L = [\oplus_{i=1}^g S/I_i]/R$ and $K = JR$. Put $H_J(L, n) = \lambda(J^n \otimes_R L)$ and $P_J(L, n)$ be the corresponding Hilbert polynomial. Then*

$$P_J(L,n) = -e_1(K)\binom{n+d-2}{d-1} + e_2(K)\binom{n+d-3}{d-2} - \dots + (-1)^d e_d(K) + \lambda(L).$$

*If $J \subseteq \operatorname{ann} L$ then*

$$P(K,n) = e_0(K)\binom{n+d-1}{d} + \lambda(L)\binom{n+d-2}{d-1} + \dots + n\lambda(L).$$

*Proof:* First we show that $\lambda(L) < \infty$. Consider the exact sequence

$$0 \longrightarrow \frac{S}{I_1 \cap \dots \cap I_g} \longrightarrow N = \frac{S}{I_1} \oplus \dots \oplus \frac{S}{I_g} \longrightarrow L \longrightarrow 0. \tag{8}$$

Let $P$ be a non-maximal prime ideal of $S$ not containing any $I_1, \dots, I_g$. Then $L_P = 0$. If there is an $i$ such that $I_i \subseteq P$, then for $j \neq i$, $I_j \nsubseteq P$. Hence

$$(S/I_1 \cap \dots \cap I_g)_P = (S/I_1 \oplus \dots \oplus S/I_g)_P = (S/I_i)_P.$$

Thus $L_P = 0$. Hence $\operatorname{Supp} L = \{\mathfrak{n}\}$. Thus $\lambda(L) < \infty$.
By the depth lemma, $\operatorname{depth} R = 1$. Thus $R$ is not Cohen-Macaulay. Tensoring (8) with $S/J^n$ we get the exact sequence

$$\longrightarrow \operatorname{Tor}_1^S(R, S/J^n) \longrightarrow \bigoplus_{i=1}^g \operatorname{Tor}_1^S(S/I_i, S/J^n) \longrightarrow \operatorname{Tor}_1^S(L, S/J^n)$$

$$\longrightarrow R/K^n \longrightarrow \bigoplus_{i=1}^g S/(I_i, J^n) \longrightarrow L/J^n L \longrightarrow 0.$$

By Lemma 15, $\operatorname{Tor}_1^S(S/I_i, S/J^n) = 0$ for all $i, n$. For large $n$, $J^n L = 0$ as $\lambda(L) < \infty$. Hence for large $n$,

$$\lambda(R/K^n) = e_0(K)\binom{n+d-1}{d} - e_1(K)\binom{n+d-2}{d-1} + \dots + (-1)^d e_d(K).$$

By (8) and additivity of $e_0(J, \_)$ we get $e_0(K) = \sum_{i=1}^g e_0(J, S/I_i)$. Hence

$$\lambda(\operatorname{Tor}_1^S(L, S/J^n)) - \lambda(L) = \sum_{i=1}^d (-1)^i e_i(K)\binom{n+d-1-i}{d-i}.$$

From the exact sequence

$$0 \longrightarrow J^n \longrightarrow S \longrightarrow S/J^n \longrightarrow 0$$

we get

$$\longrightarrow \mathrm{Tor}_1^S(J^n, L) \longrightarrow \mathrm{Tor}_1^S(S, L) \longrightarrow \mathrm{Tor}_1^S(S/J^n, L)$$
$$\longrightarrow J^n \otimes_S L \longrightarrow S \otimes L \longrightarrow L/J^n L \longrightarrow 0.$$

Hence for large $n$,

$$\lambda(\mathrm{Tor}_1^S(S/J^n, L)) = \lambda(J^n \otimes_S L) = \left[\sum_{i=1}^{d}(-1)^i e_i(K)\binom{n+d-1-i}{d-i}\right] + \lambda(L). \tag{9}$$

Since $\dim(\mathcal{R}(J)\otimes_R L) = d$ and $d \geq 2$, $e_1(K) < 0$. If $J \subseteq \operatorname{ann} L$ then by Lemma 17

$$\lambda(\mathrm{Tor}_1(L, S/J^n)) = \binom{n+d-1}{d-1}\lambda(L).$$

Substituting in (9) we get

$$\begin{aligned} & \binom{n+d-1}{d-1}\lambda(L) - \lambda(L) \\ = \quad & -e_1(K)\binom{n+d-2}{d-1} + e_2(K)\binom{n+d-3}{d-2} - \cdots + (-1)^d e_d(K). \end{aligned}$$

Using the equation,

$$\binom{n+d-1}{d-1} = 1 + \sum_{i=1}^{d-1}\binom{n+d-i-1}{d-i}$$

we obtain $e_i(K) = (-1)^i\lambda(L)$ for $i = 1, 2, \ldots, d-1$ and $e_d(K) = 0$.

□

**Example 19.** We have computed the following example using the software CoCoA which illustrates the above result. Let $S = k[x, y, z, w]$ and $I_1 = (x, y)$ and $I_2 = (z, w)$. Let $R = S/I_1 \cap I_2$ and $Q = (x + z, y + w)$ be a system of parameters of $R$. Then $e_1(Q) = -1$ and $e_2(Q) = 0$. The cocoa program which computes the example is given below.

```
Alias P:=$contrib/primary;
Use S ::= Q[x,y,z,w];
I1 :=Ideal(x,y);
I2 :=Ideal(z,w);
I :=Intersection(I1,I2);
I;
Ideal(xw,yz,yw,xz)
Dim(S/I);
2
Depth(S/I);
1
Q :=Ideal(x+z,y+w);
```

```
PS := P.PrimaryPoincare(I, Q); PS;
(3-x) / (1-x)^2
f(x)=3-x
f'(1)=e_1(Q)=-1
e_2(Q)=0
```

## 5. Solution of the Negativity Conjecture

Recently Ghezzi, Goto, Hong, Ozeki, Phoung and Vasconcelos [3] have solved the Negativity Conjecture in full generality. We sketch their solution in this section. Recall that for an $R$-module $M$ of finite Krull dimension,

$$\operatorname{Assh}(M) = \{\mathfrak{p} \in \operatorname{Ass}(M) \mid \dim R/\mathfrak{p} = \dim M\}.$$

**Lemma 20** (Goto-Nakamura, [7]). *Let $(R, \mathfrak{m})$ be a complete local ring of dimension $\geq 2$. Let $K_R$ be the canonical module of $R$ and $S = \operatorname{Hom}_R(K_R, K_R)$. Let $\phi : R \longrightarrow S$ be the map $\phi(a)(x) = ax$ for $a \in R$ and $x \in K_R$. If $\operatorname{Ass} R \subseteq \operatorname{Assh} R \cup \{\mathfrak{m}\}$, then $\ker \phi$ has finite length and $H^1_{\mathfrak{m}}(R) \simeq H^0_{\mathfrak{m}}(\operatorname{coker} \phi)$. In particular, $H^1_{\mathfrak{m}}(R)$ has finite length.*

**Lemma 21** (Goto-Nakamura, [7]). *Let $R$ be a homomorphic image of a Cohen-Macaulay local ring and assume that $\operatorname{Ass} R \subseteq \operatorname{Assh} R \cup \{\mathfrak{m}\}$. Then*

$$\mathcal{F} = \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{ht}_R \mathfrak{p} > 1 = \operatorname{depth} R_{\mathfrak{p}}, \mathfrak{p} \neq \mathfrak{m}\}$$

*is a finite set.*

**Proposition 22.** *Let $R$ be a homomorphic image of a Cohen-Macaulay local ring and assume that $\operatorname{Ass} R \subseteq \operatorname{Assh} R \cup \{\mathfrak{m}\}$. Let $\mathfrak{q}$ be a parameter ideal. Then there exists a system $a_1, \ldots, a_d$ of generators of $\mathfrak{q}$ such that $\operatorname{Ass} R/\mathfrak{q}_i \subseteq \operatorname{Assh} R/\mathfrak{q}_i \cup \{\mathfrak{m}\}$ for all $0 \leq i \leq d$ where $q_i = (a_1, a_2, \ldots, a_i)$.*

**Theorem 23** ([3]). *Let $(R, \mathfrak{m})$ be an unmixed local ring with $\dim R = d > 0$. Let $Q = (a_1, \ldots, a_d)$ be a parameter ideal in $R$ such that $e_1(Q) = 0$. Then $R$ is Cohen-Macaulay.*

*Proof:* Apply induction on $d$. Let $d = 1$. As $R$ is unmixed, it is Cohen-Macaulay. Hence $e_1(Q) = 0$. Now consider the $d = 2$ case. Since $R$ is unmixed we can choose a nonzerodivisor $a_1 \in Q$ which is a superficial element for $Q$. Let $\overline{R} = R/a_1R$. We have $e_1(Q\overline{R}) = e_1(Q) = 0$. Since $\overline{R}$ is a 1-dimensional, $e_1(Q\overline{R}) = -\lambda(H^0_{\mathfrak{m}}(\overline{R})) = 0$, which implies $e_1(Q\overline{R}) = 0$. Hence $\bar{R}$ is Cohen-Macaulay and so is $R$.
Assume that $d \geq 3$ and that our assertion holds true for $d - 1$. Then we can choose by Proposition 22, an element $x = a_1$ superficial for the parameter ideal $Q$ and $\operatorname{Ass}(R/xR) \subseteq \operatorname{Assh}(R/xR) \cup \{\mathfrak{m}\}$. Let $U$ be the unmixed component of

(0) in $B = R/xR$. Let $(0)B = \cap_{\mathfrak{p} \in \operatorname{Ass} B} \mathfrak{q}(\mathfrak{p})$. Note that

$$\begin{aligned} H^0_{\mathfrak{m}}(B) &= 0 :_B \mathfrak{m}^\infty \\ &= \bigcap_{\mathfrak{p} \in \operatorname{Ass} B} \mathfrak{q}(\mathfrak{p}) :_B \mathfrak{m}^\infty \\ &= \bigcap_{\mathfrak{p} \in \operatorname{Ass} B} \mathfrak{q}(\mathfrak{p}) :_B \mathfrak{m}^\infty \end{aligned}$$

Since $(\mathfrak{q}(\mathfrak{p}) : \mathfrak{m}^\infty) = B$ if $\sqrt{\mathfrak{q}(\mathfrak{p})} = \mathfrak{m}$ and $(\mathfrak{q}(\mathfrak{p}) : \mathfrak{m}^\infty) = \mathfrak{q}(\mathfrak{p})$ otherwise, thus $H^0_{\mathfrak{m}}(B) = U$. Hence $U$ is of finite length. Notice that the $(d-1)$-dimensional ring $B/U$ is Cohen-Macaulay by the induction hypothesis because $B/U$ is unmixed and

$$e_1(Q(B/U)) = e_1(QB) = e_1(Q) = 0.$$

Hence $H^i_{\mathfrak{m}}(B/U) = 0$ for $0 \leq i \leq d-2$. Since $H^i_{\mathfrak{m}}(B) = H^i_{\mathfrak{m}}(B/U)$ for all $i > 0$, so $H^i_{\mathfrak{m}}(B) = 0$ for $1 \leq i \leq d-2$. Consider the short exact sequence

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow B \longrightarrow 0.$$

Applying the local cohomology functor on this sequence we get the following long exact sequence of local cohomology modules.

$$0 \longrightarrow H^0_{\mathfrak{m}}(R) \longrightarrow H^0_{\mathfrak{m}}(R) \longrightarrow H^0_{\mathfrak{m}}(B) \longrightarrow H^1_{\mathfrak{m}}(R) \longrightarrow H^1_{\mathfrak{m}}(R) \longrightarrow H^1_{\mathfrak{m}}(B) \longrightarrow$$

Since $H^1_{\mathfrak{m}}(B) = 0$, $H^1_{\mathfrak{m}}(R) = xH^1_{\mathfrak{m}}(R)$. As ${H^1}_m(R)$ is finitely generated, by Nakayama's Lemma $H^1_{\mathfrak{m}}(R) = 0$. Since $R$ is unmixed $H^0_{\mathfrak{m}}(R) = 0$. From the above long exact sequence we get $H^0_{\mathfrak{m}}(B) = 0$, which implies that $B$ is Cohen-Macaulay. Hence $R$ is Cohen-Macaulay as $x$ is regular.

□

**Example 24.** Let $S = k[|x, y, z, u, v, w|]$ and $I = (x, y) \cap (z, u, v, w)$ and $R = S/I$. Then $Q = (x + z, u, y + v, w)$ is a system of parameters in $R$. By computations we have seen that $e_1(Q) = 0$ but $R$ is not Cohen-Macaulay as depth of $R$ is 1. So unmixed condition is necessary.

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